Norm Properties of C-Numerical Radii

Moshe Goldberg and E. G. Straus*

Department of Mathematics
University of California
Los Angeles, California 90024

Submitted by Alston Householder

ABSTRACT

Given $n \times n$ complex matrices A, C, the C-numerical radius of A is the nonnegative quantity

$$r_C(A) \equiv \max\{|\mathrm{tr}(CU^*AU)| : U \text{ unitary}\}.$$

For $C = \mathrm{diag}(1, 0, \ldots, 0)$ it reduces to the classical numerical radius $r(A) = \max\{|x^*Ax| : x^*x = 1\}$. We show that r_C is a generalized matrix norm if and only if C is nonscalar and $\mathrm{tr}\, C \neq 0$. Next, we consider an arbitrary generalized matrix norm and characterize all constants $\nu > 0$ for which νN is multiplicative. A technique to obtain such ν is then applied to C-numerical radii with Hermitian C. In particular we find that νr is a matrix norm if and only if $\nu \geqslant 4$.

1. INTRODUCTION

Let $\mathbf{C}_{n \times n}$ be the algebra of $n \times n$ complex matrices, and let $\mathcal{U}_n$ be its unitary group. Given $A, C \in \mathbf{C}_{n \times n}$, the C-numerical range of A is the compact set

$$W_C(A) = \{\mathrm{tr}(CU^*AU) : U \in \mathcal{U}_n\}.$$

This definition together with some properties of $W_C(A)$ were presented by the authors in [2].

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It is not hard to see (compare [2], Lemma 9), that $W_C(A)$ is invariant under unitary similarities of A or C. Hence, if C is normal with eigenvalues γ_j, we easily find that

$$W_C(A) = W_{\mathrm{diag}(\gamma_1,\dots,\gamma_n)}(A) = \left\{ \sum_{j=1}^{n} \gamma_j x_j^* A x_j : \{x_j\} \in \Lambda_n \right\}, \qquad (1.1)$$

Λ_n being the set of orthonormal bases for $\mathbf{C}_n$. In particular, for $C=\mathrm{diag}(1,0,\dots,0)$, we obtain the classical range

$$W(A) = \{x^*Ax : x^*x = 1\}.$$

Associated with the classical range is the numerical radius

$$r(A) = \max\{|z| : z \in W(A)\}.$$

Similarly, we define the *C-numerical radius* to be

$$r_C(A) = \max\{|z| : z \in W_C(A)\}.$$

The main purpose of this work is to study the norm properties of r_C. The situation is trivial for $n=1$, so without further reference *we assume throughout the paper that* $n \geqslant 2$.

We use the following standard definitions.

(i) A mapping $A \to N(A)$ is a *seminorm* on $\mathbf{C}_{n\times n}$ if for any A, $B \in \mathbf{C}_{n\times n}$ and $\alpha \in \mathbf{C}$,

$$N(A) \geqslant 0,$$

$$N(\alpha A) = |\alpha| N(A),$$

$$N(A+B) \leqslant N(A) + N(B).$$

(ii) A seminorm is a *generalized matrix norm* if it is positive definite, that is,

$$N(A) > 0 \qquad \text{for} \quad A \neq 0.$$

(iii) A generalized matrix norm is a *matrix norm* if it is (sub-) multiplicative, i.e., for all A, B,

$$N(AB) \leqslant N(A)N(B).$$

Without difficulty we obtain

THEOREM 1. *For any C, r_C is a seminorm.*

The questions of definiteness and multiplicativity are much more complicated.

In Sec. 2 we characterize those C for which r_C is positive definite. We show that r_C is a generalized matrix norm if and only if C is not scalar and $\operatorname{tr} C \neq 0$. This result agrees with the well-known fact that the classical radius r is a generalized matrix norm.

The classical radius is not multiplicative [4]. Hence, in general, a C-radius cannot be expected to be a matrix norm.

In Sec. 3 we consider arbitrary generalized matrix norms N, and characterize all positive constants ν for which νN is multiplicative. A technique of finding such *multiplicativity factors* is given by a theorem of Gastinel [1].

The above technique (aided by some combinatorial inequalities obtained in Sect. 4) is applied in Sec. 5 to find multiplicativity factors for C-numerical radii with Hermitian C. In particular we find that νr is a matrix norm if and only if $\nu \geq 4$.

2. NORM CHARACTERIZATION OF C-RADII.

THEOREM 2. *r_C is a generalized matrix norm if and only if*

$$C \text{ is nonscalar} \quad \text{and} \quad \operatorname{tr} C \neq 0. \tag{2.1}$$

In the proof we use the following three lemmas in which A, C are given $n \times n$ matrices.

LEMMA 1. *Let m be an integer with $1 \leq m < n$. If C leaves invariant all m-dimensional subspaces of $\mathbf{C}^n$, then C is scalar.*

Proof. Since $m < n$, then each one-dimensional subspace of $\mathbf{C}^n$ is an intersection of subspaces of dimension m, which by hypothesis, are fixed by C. This implies that C fixes all one-dimensional subspaces of $\mathbf{C}_n$.

Now let $\{e_j\}_{j=1}^n$ be the standard basis of $\mathbf{C}^n$. By the preceding argument, there exist scalars $\lambda_1, \ldots, \lambda_n$, μ, such that

$$Ce_j = \lambda_j e_j, \qquad 1 \leq j \leq n,$$

and

$$C\sum_{j=1}^{n} e_j = \mu \sum_{j=1}^{n} e_j.$$

Hence, $\mu\Sigma e_j = \Sigma\lambda_j e_j$, and we conclude that $\lambda_j = \mu$, $1 \leqslant j \leqslant n$. Therefore,

$$Ce_j = \mu e_j, \qquad 1 \leqslant j \leqslant n;$$

i.e., $C = \mu I$, and the lemma follows. ■

LEMMA 2. *If*

$$CU^*AU = U^*AUC \qquad \forall U \in \mathcal{U}_n,$$

then either A or C is scalar.

Proof. Suppose A is not scalar, and let us prove that C is. Let λ be an eigenvalue of A with corresponding eigenspace $\mathbf{V}_\lambda$ of dimension m. Since A is not scalar, then

$$1 \leqslant m = \dim(\mathbf{V}_\lambda) < \dim(\mathbf{C}^n) = n.$$

Now, for arbitrary $U \in \mathcal{U}_n$, U^*AU also has λ as eigenvalue with corresponding eigenspace $U^*\mathbf{V}_\lambda$. Thus, for every vector $v \in U^*\mathbf{V}_\lambda$,

$$U^*AU(Cv) = C(U^*AUv) = C(\lambda v) = \lambda(Cv).$$

It follows that

$$Cv \in U^*\mathbf{V}_\lambda \qquad \forall v \in U^*\mathbf{V}_\lambda,$$

that is, C leaves $U^*\mathbf{V}_\lambda$ invariant. Since $\dim(\mathbf{V}_\lambda) = m$ and U^* is arbitrary, we find that C leaves invariant all m-dimensional subspaces of $\mathbf{C}^n$. Hence, by Lemma 1, C is scalar and the proof is complete. ■

LEMMA 3. *If*

$$\operatorname{tr}(CU^*AU) = \textit{constant} \qquad \forall U \in \mathcal{U}_n,$$

then

$$CU^*AU = U^*AUC, \qquad \forall U \in \mathfrak{U}_n.$$

Proof. Let S be skew-Hermitian; then $e^{\theta S}$ is unitary for all real θ, and so is $Ue^{\theta S}$. By hypothesis therefore,

$$f(\theta) \equiv \operatorname{tr}\left[C(Ue^{\theta S})^* A(Ue^{\theta S})\right] = \text{constant}, \qquad \theta \in \mathbf{R};$$

and consequently,

$$\frac{d}{d\theta} f(\theta) = \frac{d}{d\theta} \operatorname{tr}(Ce^{-\theta S}U^*AUe^{\theta S})$$

$$= \operatorname{tr}(Ce^{-\theta S}U^*ASe^{\theta S} - CSe^{-\theta S}U^*AUe^{\theta S}) = 0.$$

Evaluating the derivative at $\theta = 0$, we obtain

$$\operatorname{tr}(CU^*AUS - CSU^*AU) = 0;$$

hence for all skew-Hermitian S (and all unitary U),

$$\operatorname{tr}\left[(CU^*AU - U^*AUC)S\right] = 0.$$

Since every matrix B is a linear combination of skew-Hermitians,[1] the last identity implies

$$\operatorname{tr}\left[(CU^*AU - U^*AUC)B\right] = 0 \qquad \forall B \in \mathbf{C}_{n\times n}.$$

Thus,

$$CU^*AU - U^*AUC = 0,$$

and the lemma is proven. ■

Proof of Theorem 2. By Theorem 1, it suffices to show that (2.1) holds if and only if r_C is positive definite.

[1]For example, $B = S_1 - iS_2$ with $S_1 = \frac{1}{2}(B - B^*)$, $S_2 = (i/2)(B + B^*)$.

If C is scalar, namely $C=\lambda I$, then any $A\neq 0$ with $\operatorname{tr} A=0$ gives

$$r_C(A) = |\lambda \operatorname{tr} A| = 0.$$

Also, if $\operatorname{tr} C=0$, then

$$r_C(I) = |\operatorname{tr} C| = 0.$$

Thus, violation of (2.1) implies the indefiniteness of $r_C(\cdot)$.

Conversely, let (2.1) hold. If $r_C(A)=0$, then by definition

$$\operatorname{tr}(CU^*AU^*) = 0 \qquad \forall U \in \mathcal{U}_n;$$

so by Lemma 3,

$$CU^*AU = U^*AUC \qquad \forall U \in \mathcal{U}_n.$$

By Lemma 2, therefore, either C or A is scalar, and since C is not, A is. Setting $A=\mu I$ we have

$$r_C(A) = |\mu \operatorname{tr} C| = 0,$$

and since $\operatorname{tr} C\neq 0$, then μ must vanish and the proof is established. ∎

Example 1. The k-numerical range, $1\leqslant k\leqslant n$, was defined by Halmos [3, Sec. 167] to be

$$W_k(A) = \{\operatorname{tr}(PA) : P \text{ orthonormal projection of rank } k\}.$$

We easily verify that

$$W_k(A) = W_{C_k}(A), \qquad \text{where} \quad C_k = I_k \oplus 0_{n-k}.$$

Thus, the *k-numerical radius*

$$r_k(A) = \max\{|z| : z \in W_k(A)\}$$

is a generalized matrix norm if and only if $1\leqslant k\leqslant n-1$. In particular, $r(A)=r_1(A)$ is a generalized norm, while $r_n(A)=|\operatorname{tr} A|$ is not.

3. MULTIPLICATIVITY FACTORS AND GASTINEL'S THEOREM

Given a seminorm N on $\mathbf{C}_{n\times n}$ and a constant $\nu>0$, then obviously

$$N_\nu \equiv \nu N$$

is a seminorm too. Similarly, N is definite if and only if N_ν is. In any case the new norm may or may not be multiplicative. If it is, we say that ν is a *multiplicativity factor* of N.

A characterization of multiplicativity factors for generalized matrix norms is given in Theorem 4. We first prove, however, that indefinite nontrivial seminorms have no multiplicativity factors.

THEOREM 3. *An indefinite seminorm N on $\mathbf{C}_{n\times n}$ is multiplicative if and only if $N\equiv 0$.*

Proof. The trivial semi-norm is certainly multiplicative. So let N be indefinite and multiplicative, and let us show that $N\equiv 0$.

Since N is indefinite, then $N(A)=0$ for some $A\neq 0$. Let α_{lk} be a nonvanishing entry of A, and denote by E_{ij} the matrix whose (i,j) element is 1 and the others are zero. Since

$$E_{il}AE_{kj} = \alpha_{lk}E_{ij},$$

then by multiplicativity,

$$|\alpha_{lk}|N\left(E_{ij}\right) = N\left(\alpha_{lk}E_{ij}\right) \leqslant N\left(E_{il}\right)N\left(A\right)N\left(E_{kj}\right) = 0.$$

We conclude that

$$N\left(E_{ij}\right) = 0 \qquad \forall 1 \leqslant i,j \leqslant n;$$

thus for any $B=(\beta_{ij})\in\mathbf{C}_{n\times n}$,

$$N(B) = N\left(\sum_{i,j}\beta_{ij}E_{ij}\right) \leqslant \sum_{i,j}|\beta_{ij}|N\left(E_{ij}\right) = 0,$$

and the theorem follows. ∎

THEOREM 4. *If N is a generalized matrix norm, then ν is a multiplicativity factor of N (i.e., N_ν is a matrix norm) if and only if*

$$\nu \geqslant \nu_N \equiv \max_{A,B\neq 0} \frac{N(AB)}{N(A)N(B)}.$$

Proof. We write ν_N in the form

$$\nu_N = \max\{N(AB): N(A)=N(B)=1\},$$

and use a compactness argument to conclude that ν_N is well defined. It is clear then that $\nu_N > 0$.

Now, if $\nu \geqslant \nu_N$, then

$$N_\nu(AB) = \nu N(AB) \leqslant \nu\nu_N N(A)N(B) \leqslant \nu^2 N(A)N(B) = N_\nu(A)N_\nu(B);$$

hence N is multiplicative.

Conversely, if ν satisfies $0<\nu<\nu_N$, we can find matrices A, B such that $\nu N(A)N(B)<N(AB)$. Thus we have

$$N_\nu(AB) = \nu N(AB) > \nu^2 N(A)N(B) = N_\nu(A)N_\nu(B),$$

and the proof is complete. ∎

As an immediate consequence we have established

COROLLARY 1. *A generalized matrix norm N is a matrix norm if and only if $\nu_N \leqslant 1$.*

In practice, Theorem 4 offers limited help, since in general ν_N is not easily evaluated. In the case of C-numerical radii, we were unable to find the optimal factor except for the classical radius.

An alternative way of finding multiplicativity factors is suggested by the following, somewhat stronger version of a theorem by Gastinel, [1].

THEOREM 5. *Let N be seminorm, M a matrix norm, and $\eta \geqslant \xi > 0$ constants such that*

$$\xi M(A) \leqslant N(A) \leqslant \eta M(A) \qquad \forall A \in \mathbf{C}_{n\times n}. \tag{3.1}$$

Then

(i) *N is a generalized matrix norm.*
(ii) *For any $\nu \geqslant \eta/\xi^2, N_\nu$ is a matrix norm.*
(iii) *If $\eta/\xi^2 \leqslant 1$, then N is a matrix norm.*

Proof. Part (i) is trivial, and for part (ii) we should merely note that

$$N_\nu(AB) = \nu N(AB) \leqslant \nu\eta M(AB) \leqslant \nu\eta M(A)M(B)$$

$$\leqslant \frac{\nu\eta}{\xi^2} N(A)N(B) \leqslant \nu^2 N(A)N(B) = N_\nu(A)N_\nu(B).$$

Part (iii) then follows. ■

We recall, of course, that any two norms on $\mathbf{C}_{n\times n}$ are equivalent. Thus if N of Theorem 5 is known to be a generalized matrix norm, then (3.1) always holds for suitable constants $\eta \geqslant \xi > 0$.

In Sec. 5 we use Theorem 5 to obtain multiplicativity factors for C-numerical radii with Hermitian C.

4. SOME COMBINATORIAL INEQUALITIES

Let α_j, γ_j, $1 \leqslant j \leqslant n$, be scalars, and consider the set

$$\mathcal{S}_\gamma(\alpha) = \left\{ \sum_{j=1}^{n} \gamma_j \alpha_{\sigma(j)} : \sigma \in S_n \right\},$$

S_n being the symmetric group. In this section we study bounds for the radius of $\mathcal{S}_\gamma(\alpha)$,

$$R_\gamma(\alpha) = \max\{|z| : z \in \mathcal{S}_\gamma(\alpha)\}.$$

A general remark is that all the quantities involved are invariant under rearrangements of the α_i and the γ_j, and under rotations of the form

$$(\alpha_1, \ldots, \alpha_n) \to e^{i\varphi}(\alpha_1, \ldots, \alpha_n), (\gamma_1, \ldots, \gamma_n) \to e^{i\psi}(\gamma_1, \ldots, \gamma_n),$$

which include, of course, change of sign. This fact will be repeatedly used in the proof of the following results.

LEMMA 4. *For any* $\alpha_j, \gamma_j \in \mathbf{C}$,

$$R_\gamma(\alpha) \geqslant \frac{1}{n}\left|\sum_{j=1}^{n} \alpha_j\right|\left|\sum_{j=1}^{n} \gamma_i\right|.$$

Proof. Let τ^i, $i=1,2,\ldots,n$, be the powers of a nontrivial cyclic permutation in S_n. Since

$$\sum_{j=1}^{n} \gamma_j \alpha_{\tau^i(j)} \in \mathcal{S}_\gamma(\alpha), \qquad 1 \leqslant i \leqslant n,$$

then

$$R_\gamma(\alpha) \geqslant \left|\frac{1}{n}\sum_{i=1}^{n}\left(\sum_{j=1}^{n}\gamma_j\alpha_{\tau^i(j)}\right)\right|$$

$$= \left|\frac{1}{n}\sum_j \gamma_j \sum_i \alpha_{\tau^i(j)}\right| = \left|\frac{1}{n}\sum_j \gamma_j \sum_i \alpha_i\right|,$$

and the lemma holds. ∎

LEMMA 5. *If* $\alpha_i \in \mathbf{R}$, $\gamma_j \in \mathbf{C}$, $1 \leqslant j \leqslant n$, *then*

$$R_\gamma(\alpha) \geqslant \frac{1}{2}\max_{i,j}|\alpha_i - \alpha_j| \max_{i,j}|\gamma_i - \gamma_j|.$$

Proof. Setting

$$\gamma_j = \lambda_j + i\mu_j, \qquad \lambda_j, \mu_j \in \mathbf{R},$$

we have

$$R_\gamma(\alpha) = \max_{\sigma \in S_n}\left|\sum_j \lambda_j \alpha_{\sigma(j)} + i\sum_j \mu_j \alpha_{\sigma(j)}\right|$$

$$\geqslant \max_{\sigma \in S_n}\left|\sum_j \lambda_j \alpha_{\sigma(j)}\right| = R_\lambda(\alpha)$$

Now, if the γ_j are equal, then the result is trivial; so by rotating and rearranging the γ_j, we may assume that

$$\max|\gamma_i - \gamma_j| = \gamma_1 - \gamma_n > 0.$$

It follows that

$$\lambda_1 - \lambda_n = \gamma_1 - \gamma_n = \max_{i,j}|\gamma_i - \gamma_j| \geqslant \max_{i,j}|\lambda_i - \lambda_j|.$$

Thus

$$\lambda_1 \geqslant \lambda_j \geqslant \lambda_n, \qquad 2 \leqslant j \leqslant n-1,$$

so we may assume that

$$\lambda_1 \geqslant \lambda_2 \geqslant \dots \geqslant \lambda_n.$$

We may also assume that

$$\alpha_1 \geqslant \alpha_2 \geqslant \dots \geqslant \alpha_n.$$

Hence, observing that

$$s_1 = \sum \lambda_j \alpha_j, \qquad s_2 = \sum \lambda_j \alpha_{n-j}$$

are two points in $\mathcal{S}_\gamma(\alpha)$, we have

$$\begin{aligned} R_\lambda(\alpha) &\geqslant \tfrac{1}{2}|s_1 - s_2| = \tfrac{1}{2}|\lambda_1(\alpha_1 - \alpha_n) + \lambda_2(\alpha_2 - \alpha_{n-1}) + \dots + \lambda_n(\alpha_n - \alpha_1)| \\ &= \tfrac{1}{2}|(\lambda_1 - \lambda_n)(\alpha_1 - \alpha_n) + (\lambda_2 - \lambda_{n-1})(\alpha_2 - \alpha_{n-1}) \\ &\qquad + \dots + (\lambda_{[n/2]} - \lambda_{[n/2]+1})(\alpha_{[n/2]} - \alpha_{[n/2]+1})| \\ &\geqslant \tfrac{1}{2}(\lambda_1 - \lambda_n)(\alpha_1 - \alpha_n) = \tfrac{1}{2}\max|\gamma_i - \gamma_j|\max|\alpha_i - \alpha_j|, \end{aligned}$$

and the lemma follows. ■

We are interested now in obtaining constants K_γ, which may depend on the γ_j but not on the α_j, such that

$$R_\gamma(\alpha) \geqslant K_\gamma \max|\alpha_j| \qquad \forall \alpha_1,\dots,\alpha_n \in \mathbf{R}. \tag{4.1}$$

THEOREM 6. *For given $\gamma_j \in \mathbf{C}$, $1 \leqslant j \leqslant n$, there exists a constant $K_\gamma > 0$ which satisfies* (4.1) *if and only if*

$$\gamma_j \textit{ are not all equal} \quad \textit{and} \quad \sum_j \gamma_j \neq 0. \tag{4.2}$$

If (4.2) *holds, then* (4.1) *is satisfied by the positive constant*

$$K_\gamma = \frac{|\sum_j \gamma_j| \cdot \max_{i,j} |\gamma_i - \gamma_j|}{2|\sum_j \gamma_j| + \max_{i,j} |\gamma_i - \gamma_j|}. \tag{4.3}$$

Proof. Suppose (4.2) is violated. If the γ_j are equal, we choose α_j not all equal, with $\Sigma\alpha_j = 0$; if $\Sigma\gamma_j = 0$, we take $\alpha_j = 1$, $1 \leqslant j \leqslant n$. In both cases $R_\gamma(\alpha) = 0$ but $\max|\alpha_j| > 0$; hence no positive K_γ satisfies (4.1).

Conversely, let (4.2) hold, and let K_γ be the constant specified in (4.3). We may assume that

$$\alpha_1 \geqslant \dots \geqslant \alpha_n,$$

where in fact, by change of sign if necessary, it suffices to consider the cases

$$\alpha_1 \geqslant \dots \geqslant \alpha_n \geqslant 0, \tag{4.4a}$$

and that

$$\alpha_1 \geqslant \dots \geqslant \alpha_k \geqslant 0 > \alpha_{k+1} \geqslant \dots \geqslant \alpha_n \qquad \text{with} \quad \max|\alpha_j| = \alpha_1. \tag{4.4b}$$

In the case (4.4a) we write $\alpha_n = \theta\alpha_1$, $0 \leqslant \theta \leqslant 1$, and use Lemmas 4 and 5 to obtain, respectively,

$$R_\gamma(\alpha) \geqslant \frac{1}{n}|\Sigma\alpha_j||\Sigma\gamma_j| > \alpha_n|\Sigma\gamma_j| = \theta\alpha_1|\Sigma\gamma_j| = \theta|\Sigma\gamma_j|\max|\alpha_j|$$

and

$$R_\gamma(\alpha) \geqslant \tfrac{1}{2}\max|\alpha_i - \alpha_j|\max|\gamma_i - \gamma_j| = \tfrac{1}{2}(\alpha_1 - \alpha_n)\max|\gamma_i - \gamma_j|$$
$$\geqslant \tfrac{1}{2}(1-\theta)\max|\gamma_i - \gamma_j|\max|\alpha_j|.$$

We thus find that

$$R_\gamma(\alpha) \geqslant \max\left\{\theta\left|\sum \gamma_j\right|, \tfrac{1}{2}(1-\theta)\max|\gamma_i - \gamma_j|\right\}\cdot\max|\alpha_j|.$$

The expressions in the above braces are functions of θ which describe straight lines with opposite slopes and intersection value K_γ. Hence, for any θ

$$\max\left\{\theta\left|\sum \gamma_j\right|, \tfrac{1}{2}(1-\theta)\max|\gamma_i - \gamma_j|\right\} \geqslant K_\gamma,$$

and (4.1) follows.

In the case (4.4b) we use Lemma 5 to find that

$$R_\gamma(\alpha) \geqslant \tfrac{1}{2}(\alpha_1 - \alpha_n)\max|\gamma_1 - \gamma_n| > \tfrac{1}{2}\max|\gamma_i - \gamma_j|\max|\alpha_j|.$$

Since

$$\tfrac{1}{2}\max|\gamma_i - \gamma_j| > K_\gamma,$$

then (4.1) holds again, and the theorem is proven. ∎

The above result can be improved for certain classes of γ_j.

THEOREM 7. *If γ_j, $1 \leqslant j \leqslant n$, are complex scalars of the same argument, then* (4.1) *holds with*

$$K_\gamma = \tfrac{1}{2}\max_{i,j}|\gamma_i - \gamma_j|. \tag{4.5}$$

Proof. By change of argument and rearrangement we may assume that

$$\gamma_1 \geqslant \dots \geqslant \gamma_n \geqslant 0,$$

and that the α_j satisfy (4.4a) or (4.4b).
For (4.4a) we have

$$R_\gamma(\alpha) = \sum \gamma_j \alpha_j \geqslant \gamma_1 \alpha_1 \geqslant \tfrac{1}{2}(\gamma_1 - \gamma_n)\alpha_1;$$

and for (4.4b), Lemma 5 yields

$$R_\gamma(\alpha) \geqslant \tfrac{1}{2}(\gamma_1 - \gamma_n)(\alpha_1 - \alpha_n) > \tfrac{1}{2}(\gamma_1 - \gamma_n)\alpha_1.$$

Thus,

$$R_\gamma(\alpha) \geqslant \tfrac{1}{2}\max|\gamma_i - \gamma_j|\max|\alpha_j|,$$

and the proof is complete. ∎

Indeed, comparing K_γ of (4.5) with K_γ of (4.3), we realize that for the relevant γ_j, Theorem 7 provides a tighter lower bound for $R_\gamma(\alpha)$ than Theorem 6.

Note that the K_γ in (4.3) and (4.5) are independent of n.

5. MULTIPLICATIVE HERMITIAN RADII

As indicated previously, the purpose of this section is to obtain multiplicativity factors for C-numerical radii with Hermitian C.

LEMMA 6. *Let A, C be normal matrices with eigenvalues α_j and γ_j, respectively. Then*

$$r_C(A) = R_\gamma(\alpha).$$

Proof. Obviously, it suffices to show that

$$\operatorname{conv} W_C(A) = \operatorname{conv} \mathcal{S}_\gamma(\alpha).$$

Since $W_C(A)$ is invariant under unitary similarities of A and C, and since A and C are normal, then by (1.1),

$$W_C(A) = \left\{ \sum_{j=1}^{n} \gamma_j x_j^* \operatorname{diag}(\alpha_1, \ldots, \alpha_n) x_j : \{x_j\} \in \Lambda_n \right\}.$$

Thus, using the standard basis $\{e_j\}$, we find that every point in $\mathcal{S}_\gamma(\alpha)$ satisfies

$$\sum_j \gamma_j \alpha_{\sigma(j)} = \sum \gamma_j e^*_{\sigma(j)} \operatorname{diag}(\alpha_1, \ldots, \alpha_n) e_{\sigma(j)} \in W_C(A),$$

which gives us

$$\mathcal{S}_\gamma(\alpha) \subseteq W_C(A).$$

Conversely, take an arbitrary point,

$$\sum_j \gamma_j x_j^* \operatorname{diag}(\alpha_1, \ldots, \alpha_n) x_j \in W_C(A).$$

Since $x_j = (\chi_{j1}, \ldots, \chi_{jn})^T$, $1 \leqslant j \leqslant n$, is an orthonormal basis, then $X \equiv [|\chi_{jk}|^2]$ is a doubly stochastic matrix. Doubly stochastic matrices are convex combinations of permutation matrices P_σ. Thus writing $X = \Sigma_\sigma \lambda_\sigma P_\sigma$ and

$$a \equiv (\alpha_1, \ldots, \alpha_n)^T, \qquad c \equiv (\gamma_1, \ldots, \gamma_n)^T,$$

we have

$$\begin{aligned}
\sum_j \gamma_j x_j^* \operatorname{diag}(\alpha_1, \ldots, \alpha_n) x_j &= \sum_{j,k} \gamma_j |\chi_{jk}|^2 \alpha_k = c^T X a \\
&= c^T \Bigg[\sum_{\sigma \in S_n} \lambda \alpha_0 P_\sigma \Bigg] a = \sum_\sigma \lambda_\sigma \big(c^T P_\sigma a \big) \\
&= \sum_\sigma \lambda_\sigma \Bigg[\sum_j \gamma_j \alpha_{\sigma(j)} \Bigg] \in \operatorname{conv} \mathcal{S}_\gamma(\alpha).
\end{aligned}$$

This yields

$$W_C(A) \subseteq \operatorname{conv} \mathcal{S}_\gamma(\alpha),$$

and the lemma follows. ∎

LEMMA 7. *Let C be normal with eigenvalues γ_j, let K_γ satisfy* (4.1), *and* *let*

$$\|A\|_2 \equiv \max\{(x^*A^*Ax)^{1/2}: x^*x=1\}$$

denote the spectral norm of A. Then

$$r_C(H) \geqslant K_\gamma \|H\|_2 \qquad \forall \text{ Hermitian } H \in \mathbf{C}_{n\times n}.$$

Proof. For Hermitian H with eigenvalues α_j, we know that

$$\|H\|_2 = \max|\alpha_j|.$$

Since the α_j are real, we may use (4.1), and by Lemma 6

$$r_C(H) = R_\gamma(\alpha) \geqslant K_\gamma \max|\alpha_j| = K_\gamma \|H\|_2.$$

LEMMA 8. *If C is Hermitian, then $r_C(A) = r_C(A^*)$.*

Proof.

$$r_C(A) = \max_U |\mathrm{tr}(CU^*AU)| = \max_U |\mathrm{tr}(CU^*AU)^*|$$

$$= \max_U |\mathrm{tr}(U^*A^*UC)| = r_C(A^*).$$

LEMMA 9. *If C is Hermitian with eigenvalues γ_j, and if K_γ satisfies (4.1), then*

$$r_C(A) \geqslant \tfrac{1}{2}K_\gamma \|A\|_2 \qquad \forall A \in \mathbf{C}_{n\times n}.$$

Proof. We write $A = \frac{1}{2}(H_1 - iH_2)$, where

$$H_1 = A + A^*, \qquad H_2 = i(A - A^*)$$

are Hermitian. By Lemmas 7 and 8, and by Theorem 1,

$$\tfrac{1}{2}K_\gamma\|A\|_2 = \tfrac{1}{4}K_\gamma\|H_1 - iH_2\|_2 \leqslant \tfrac{1}{4}K_\gamma[\|H_1\|_2 + \|H_2\|_2] \leqslant \tfrac{1}{4}[r_C(H_1) + r_C(H_2)]$$
$$= \tfrac{1}{4}[r_C(A+A^*) + r_C(A-iA^*)] \leqslant \tfrac{1}{2}[r_C(A) + r_C(A^*)] = r_C(A),$$

and the proof is complete. ■

LEMMA 10. *If C is normal with eigenvalues γ_j, then*

$$r_C(A) \leqslant \sum_j |\gamma_j|\,\|A\|_2 \qquad \forall A \in \mathbf{C}_{n\times n}.$$

Proof. By (1.1) we have

$$r_C(A) = \max\left\{\left|\sum_j \gamma_j x_j^* A x_j\right| : \{x_j\} \in \Lambda_n\right\};$$

and since $|x^*Ax| \leqslant \|A\|_2$ for any unit vector x, the lemma follows.

THEOREM 8. *Let C be Hermitian, nonscalar, with* $\operatorname{tr} C \neq 0$ *and eigenvalues γ_j. Then, for any ν with*

$$\nu \geqslant 4\sum_j |\gamma_j| \left[\frac{2|\operatorname{tr} C| + \max_{i,j}|\gamma_i - \gamma_j|}{|\operatorname{tr} C| \cdot \max_{i,j}|\gamma_i - \gamma_j|}\right]^2,$$

the (Hermitian) numerical radius $\nu r_C \equiv r_{\nu C}$ is a matrix norm.

Proof. Since C is nonscalar, the γ_j are not all equal; and since $\operatorname{tr} C \neq 0$, then $\sum \gamma_j \neq 0$. Thus, by Theorem 6, the inequality in (4.1) is satisfied by the *positive* constant K_γ of (4.3). By Lemmas 9 and 10 we have therefore,

$$\frac{1}{2}\,\frac{|\sum \gamma_j| \cdot \max|\gamma_i - \gamma_j|}{2|\sum \gamma_j| + \max|\gamma_i - \gamma_j|}\|A\|_2 \leqslant r_C(A) \leqslant \sum |\gamma_j|\,\|A\|_2 \qquad \forall A \in \mathbf{C}_{n\times n},$$

and Theorem 5 completes the proof. ■

For Hermitian definite C, we improve Theorem 6 as follows.

THEOREM 9. *Let C be Hermitian nonnegative (nonpositive) definite. If C is nonscalar with eigenvalues γ_j, then for every ν with*

$$\nu \geqslant \frac{16|\operatorname{tr} C|}{\max_{i,j}|\gamma_i-\gamma_j|^2},$$

$\nu r_C \equiv r_{\nu C}$ is a matrix norm.

Proof. Since C is Hermitian definite, the γ_j are of the same sign and by Theorem 7, K_γ of (4.5) satisfies (4.1). Lemmas 9 and 10 yield now,

$$\tfrac{1}{4}\max|\gamma_i-\gamma_j|\,\|A\|_2 \leqslant r_C(A) \leqslant \sum|\gamma_j|\,\|A\|_2 = |\operatorname{tr} C|\,\|A\|_2 \qquad \forall A \in \mathbf{C}_{n\times n}. \tag{5.1}$$

Since C is nonscalar, the γ_j are not all equal; so $\max|\gamma_i-\gamma_j|>0$, and Theorem 5 completes the proof. ■

EXAMPLE 2. We recall the definition of the k-numerical radius r_k. By Theorem 7, we find that νr_k, $1\leqslant k\leqslant n-1$, is a matrix norm if $\nu\geqslant 16k$.

Example 2 implies that $\nu\geqslant 16$ is a multiplicativity factor for the classical radius r. The optimal factor, ν_r, is given in the following result.

THEOREM 10. *νr is a matrix norm if and only if $\nu\geqslant 4$; that is, $\nu_r=4$.*

Proof. It is well known (e.g., [3, Sec. 162]) that

$$\tfrac{1}{2}\|A\|_2 \leqslant r(A) \leqslant \|A\|_2 \qquad \forall A\in\mathbf{C}_{n\times n}.$$

Thus, by Theorem 5, $\nu\geqslant 4$ is a multiplicativity factor for r, and by Theorem 4, $\nu_r\leqslant 4$.

To show that $\nu_r\geqslant 4$, consider the matrices

$$A=\begin{pmatrix}0&1\\0&0\end{pmatrix}\oplus 0_{n-2},\qquad B=\begin{pmatrix}0&0\\1&0\end{pmatrix}\oplus 0_{n-2}.$$

A simple calculation shows that

$$r(A) = r(B) = \tfrac{1}{2}, \qquad r(AB) = 1.$$

Hence $r_\nu = \nu r$ satisfies

$$r_\nu(AB) \leqslant r_\nu(A) r_\nu(B)$$

if and only if $\nu \geqslant 4$, and the theorem follows. ∎

The proof of Theorem 10 is essentially given in [3, Sec. 176].

NOTE ADDED IN PROOF. Theorems 6 and 7 hold also when $\alpha_1, \ldots, \alpha_n$ are arbitrary vectors in any normed vector space over the complex field. For proof, see our article, "Combinatorial inequalities, matrix norms, and generalized numerical radii", *General Inequalities 2* (Proceedings of the International Conference, Mathematical Research Institute, Oberwolfach, 1978), Birkhäuser Verlag, Basel, (to appear).

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